# MATH 245 F20, Exam 1 Solutions

1. Let $b, c$ be odd integers. Without using theorems, prove that $b(c-2)$ is odd.

Since $b, c$ are odd, there exist integers $y, z$ with $b = 2y+1, c = 2z+1$. We calculate $b(c-2) = (2y+1)(2z-1) = 4yz - 2y + 2z - 1 = 4yz - 2y + 2z - 2 + 1 = 2(2yz - y + z - 1) + 1$. Since $y, z$ are integers, so is $2yz - y + z - 1$. Hence $b(c-2)$ is odd, being the sum of 1 with twice an integer.

2. Prove or disprove: For all propositions $p, q$, the proposition $(p \uparrow q) \downarrow (p \leftrightarrow q)$ is a contradiction.

We look at the truth table at right, and see that the last column is all $F$. Hence $(p \uparrow q) \downarrow (p \leftrightarrow q) \equiv F$, and therefore $(p \uparrow q) \downarrow (p \leftrightarrow q)$ is a contradiction.

| $p$ | $q$ | $p \uparrow q$ | $p \leftrightarrow q$ | $(p \uparrow q) \downarrow (p \leftrightarrow q)$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $F$ |

3. Let $p, q, r, s$ be propositions. Prove that $p \vee q, q \wedge r, p \to s \vdash q \vee s$.

We begin by assuming that $p \vee q, q \wedge r$, and $p \to s$ are all true.
SOLUTION 1: We only need the hypothesis $q \wedge r$. By simplification, $q$. By addition, $q \vee s$.

SOLUTION 2: We have two cases, based on $p \vee q$. Case 1: If $p$ is true, we apply modus ponens to $p \to s$ to get $s$. By addition, $q \vee s$. Case 2: If instead $q$ is true, we directly apply addition to get $q \vee s$. In both cases $q \vee s$ holds.

SOLUTION 3: It is also possible to do this with a huge truth table (16 rows!). NOT RECOMMENDED

4. Prove the following without truth tables: For any propositions $p, q, r, s$, we have $p \to q, q \to r, r \to s \vdash p \to s$.

We begin by assuming that $p \to q, q \to r, r \to s$ are all true.
We consider two cases: $q$ might be T or F. If $q$ is F, then by modus tollens with $p \to q$, we have $\neg p$. By addition, $s \vee \neg p$. If instead $q$ is T, then by modus ponens with $q \to r$, we have $q$. By modus ponens with $r \to s$, we have $s$. By addition, $s \vee \neg p$.

In both cases, we get $s \vee \neg p$. Finally, by conditional interpretation, we get $p \to s$.

It is also possible to do this using different cases, such as $p$ being T or F.

5. Let $x \in \mathbb{R}$. Prove that if $x^2$ is irrational, then $x$ is irrational.

We use a contrapositive proof. Assume that $x$ is rational. Then there are integers $a, b$ with $b \neq 0$ and $x = \frac{a}{b}$. We have $x^2 = \frac{a^2}{b^2}$. Note that $a^2, b^2$ are integers (since $a, b$ are), and $b^2 \neq 0$ (since $b \neq 0$). Hence $x^2$ is rational.

6. Fix our domain to be $\mathbb{Z}$ for all variables. Simplify the following proposition as much as possible (where nothing is negated): $\neg\ \forall x\ \forall y\ \exists z\ (x < y) \to (x < z \leq y)$.

We first pull the negation inside the quantifiers: $\exists x\ \exists y\ \forall z\ \neg\ ((x < y) \to (x < z \leq y))$.
We now apply Theorem 2.16 to get: $\exists x\ \exists y\ \forall z\ (x < y) \wedge \neg(x < z \leq y)$.
We interpret the double inequality (see p.11) to get: $\exists x\ \exists y\ \forall z\ (x < y) \wedge \neg((x < z) \wedge (z \leq y))$.
We apply De Morgan's Law (for propositions) to get: $\exists x\ \exists y\ \forall z\ (x < y) \wedge ((\neg(x < z)) \vee (\neg(z \leq y)))$.
We now simplify to get our answer: $\exists x\ \exists y\ \forall z\ (x < y) \wedge ((x \geq z) \vee (z > y))$.
NOTE: $((x \geq z) \vee (z > y))$ cannot be combined to a double inequality, but it is possible to use distributivity to get the alternative answer $\exists x\ \exists y\ \forall z\ (z \leq x < y) \vee (x < y < z)$.

7. Prove or disprove this proposition: $\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z},\ (x \neq y) \wedge (y|x)$.

The statement is true, and we will supply a direct proof. Let $x \in \mathbb{Z}$ be arbitrary. We have two cases, based on whether $x = 0$. NOTE: it is not possible to pick a single $y$ that works for every $x$.

If $x = 0$, choose $y = 5$. We have $x \neq y$ and $0 = (0)(5)$, so $y|x$.

If $x \neq 0$, choose $y = -x$. We have $x \neq y$, since otherwise $x = y = -x$ and so $x = -x$ but $x \neq 0$. Also $x = (-1)(y)$, so $y|x$.